MATB41 Week 7 Notes

1. Limit Definition of Partial Derivatives:

E.g.

$$f(x,y) = \begin{cases} \dfrac{x^3+x^4-y^3}{x^2+y^2} & \text{, if } (x,y) \neq (0,0) \\ 0 & \text{, if } (x,y) = (0,0) \end{cases}$$

Find $\frac{\partial f}{\partial x}(0,0)$.

Solution:

We have to use the limit definition of partial derivatives and the fact that $f(0,0)=0$ to solve this.

$$f(h,0) = \frac{h^3+h^4-0^3}{h^2+0^2}$$
$$= h+h^2$$

Then,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{h+h^2-0}{h}$$
$$= \lim_{h\to 0} 1+h$$
$$= 1$$

Note: This does NOT imply that $f(x,y)$ is cont at $(0,0)$.

2. Iterated Partial Derivatives:

- Definition: A function whose partial derivatives exist and are cont is said to be of class C^1.

  In general, a function f is of class C^n is if f has cont iterated partial derivatives of the n^{th} order.

- Examples:

1. Let $f(x,y) = x^2y^3$

1st Order Derivatives

$$\frac{\partial f}{\partial x} = 2xy^3$$

$$\frac{\partial f}{\partial y} = 3x^2y^2$$

2nd Order Derivatives/ Iterated Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6x^2y$$

Mixed Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6xy^2$$

2. Let $f(x,y) = xy + (x+2y)^2$.
Find all second partial derivatives of $f(x,y)$.

Solution:

1. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\, y + 2(x+2y) = \underline{2}$

2. $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\, x + 4(x+2y) = \underline{8}$

3. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}\, x + 4(x+2y) = \underline{5}$

4. $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}\, y + 2(x+2y) = \underline{5}$

- Given $f(x,y)$,
$\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial x}$ are all iterated partial derivatives.

However, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are also called mixed partial derivatives.

- Thm of Equality of Mixed Partials:

If $f(x,y)$ is of class C^2, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

If $f(x,y,z)$ is of class C^3, then

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2}$$

and

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial y}$$

The order does not matter.

In general, suppose f is a function of n variables defined on an open subset U of R^n. Suppose all mixed partial derivatives with a certain number of differentiations in each variable exist and are continuous on U. Then, all mixed partials are continuous.

Example:

Let (x,y) be Cartesian Coordinates.
Let (r,θ) be polar coordinates.

a) Let $z = f(x,y)$. Express $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

Recall that $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x} \cos\theta + \frac{\partial z}{\partial y} \sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

b) Express $\frac{\partial^2 z}{\partial r^2}$ in terms of cartesian coordinates.

Solution:

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)$$

3. Equations that use partial derivatives:

1. Heating Equation:

$$\frac{\partial T}{\partial t} = a^2 \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

2. Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

3. Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Note: The solutions to Laplace Equation are called Harmonic Functions.

4. Taylor's Theorem:

- Definition:

For a smooth function $f: R \to R$, the Taylor series of f at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

- Partial sum of Taylor Series:

The partial sum of the Taylor Series for a function f, denoted by $T_n(x)$, is a polynominal of degree n called the n^{th}-degree Taylor polynominal of f at a, $n = 0, 1, 2, \dots$

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Properties:

1. $T_n(x)$ is a polynominal in powers of $(x-a)$.
2. The line $T_1(x)$ is tangent to $y=f(x)$ at the point $(a, f(a))$. $y=T_1(x)$ and $y=f(x)$ have the same slope, $f'(a)$, at this point.

3. The second derivative measures the way the curve $y = f(x)$ is bending as it passes through $(a, f(a))$. That is, $f''(a)$ is the concavity of $y = f(x)$ at $(a, f(a))$. $y = T_2(x)$ and $y = f(x)$ have the same concavity at this point.

4. $y = T_3(x)$ and $y = f(x)$ have the same rate of change of concavity at $(a, f(a))$.

5. The larger n is, the more closely the n^{th} degree Taylor polynominal will approximate $f(x)$ for x near a.

6. If $f(x) = T_n(x) + R_n$ where T_n is the n'th-degree Taylor polynominal of f at a and $\lim_{n \to \infty} R_n(x) = 0$ for $|x-a| < R$, then

f is equal to the sum of its Taylor Series on the interval $|x-a| < R$.

$R_n(x) = f(x) - T_n(x)$ is called the n^{th} degree remainder for $f(x)$ at $x=a$. It is the error made if $f(x)$ is replaced with the approximation $T_n(x)$.

If $\left| f^{(n+1)}(x) \right| \leq M$ for $|h| < R$, then

$$\left| R_n(x) \right| = \left| \int_a^x \frac{(x-\tau)^n}{n!} f^{n+1}(\tau)\, d\tau \right| \leq \frac{M}{(n+1)!} |h|^{n+1}$$

5. Taylor Series of Several Functions:

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

2. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

These 3 series converge for all $x \in \mathbb{R}$.

4. $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + \frac{a}{1!} x + \frac{a(a-1)}{2!} x^2 + \dots$

5. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{x^n}{n}\right)$

6. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{2n+1}\right)$

The last 3 series converge if $|x| < 1$.

- Geometric Series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

If $|x| < 1$, the geometric series converges to $\frac{1}{1-x}$.

6. Thm: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be a class of C^{n+1}.
Then, the n^{th} order Taylor series of function f at $x = x_0$ is

$$f(x_0+h) = f(x_0) + \sum_{i_1=1}^{n} h_{i_1} \frac{\partial f}{\partial x_{i_1}}(x_0) +$$

$$\frac{1}{2!} \sum_{i_1, i_2 = 1}^{n} h_{i_1} h_{i_2} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x_0) + \dots +$$

$$\frac{1}{n!} \sum_{i_1, i_2, \dots i_n = 1}^{n} h_{i_1} h_{i_2} \dots h_{i_n} \frac{\partial^n f}{\partial x_{i_1} \dots \partial x_{i_n}} + R_n(x_0, h)$$

$h_{i_1} = (x_{i_1} - x_0)$
$h_{i_2} = (x_{i_2} - x_0)$
$h_{i_n} = (x_{i_n} - x_0)$

$$R_n(x_0, h) = \frac{1}{(n+1)!} \sum_{i_1, i_2, \dots i_{n+1}}^{n} h_{i_1} \dots h_{i_{n+1}} \frac{\partial^{n+f}}{\partial x_1 \dots \partial x_{n+1}}(c_{i_1, \dots i_{n+1}})$$

where $c_{i_1, \dots i_{n+1}}$ is a point on the line joining x_0 and x_0+h and $\frac{R_n(x_0,h)}{\|h\|^n} \to 0$.

- Examples:
1. First-Order Taylor Formula:
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^2. Then,
$f(x_0+h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + R_1(x_0, h)$

2. Second-Order Taylor Formula:
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^3. Then,
$f(x_0+h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2!} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(x_0, h)$

3. The Second-Order Taylor Formula at $(0,0)$ is

$$T_2(x,y) = f(0,0) + \left[x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) \right] +$$

$$\frac{1}{2!}\left[x^2\frac{\partial^2 f}{\partial x^2}(0,0) + 2xy\frac{\partial^2 f}{\partial x \partial y}(0,0) + y^2\frac{\partial^2 f}{\partial y^2}(0,0) \right]$$

where $f(x,y) = T_2(x,y) + R_2(0,h)$ if $\frac{R_2(0,h)}{\|h\|} \to 0$

and $h = (x-0, y-0)$ and $R_2(0,h) = \frac{1}{3!}\sum_{i,j,k=1}^{2} h_i h_j h_k$

$$\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(c_{ijk}),$$

where c_{ijk} is a point on the line joining $(0,0)$ and (x,y).

4. Write the second order Taylor Formula for the function $f(x,y) = \frac{1}{1-x-y^2}$ around the point $(x,y) = (0,0)$.

Solution:

$f(0,0) = 1$

$\frac{\partial f}{\partial x} = \frac{1}{(1-x-y^2)^2}$ $\quad \frac{\partial f}{\partial x}(0,0) = 1$

$\frac{\partial f}{\partial y} = \frac{2y}{(1-x-y^2)^2}$ $\quad \frac{\partial f}{\partial y}(0,0) = 0$

$\frac{\partial^2 f}{\partial x^2} = \frac{2}{(1-x-y^2)^3}$ $\quad \frac{\partial^2 f}{\partial x^2}(0,0) = 2$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{4y}{(1-x-y^2)^3}$ $\quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2(1-x-y^2)+8y^3}{(1-x-y^2)^3} \qquad \frac{\partial^2 f}{\partial y^2}(0,0)=2$$

$$T_3(x,y) = 1+x+2x^2+2y^2$$

5. Write the first four terms of the Taylor Series for the function $f(x,y) = \frac{1}{1-x-y^2}$ around the point $(x,y)=(0,0)$.

Soln:
The first term is always $f(x_0, y_0)$ where (x_0, y_0) is the given point.

Furthermore, we know that the Geometric Series has $\frac{1}{1-r} = 1+r+r^2+r^3+\ldots$ for $|r|<1$. If we rewrite the eqn as $\frac{1}{1-(x+y^2)}$ and sub $x+y^2$ for r, we get $1+(x+y^2)+(x+y^2)^2+(x+y^2)^3$ as our first four terms.

6. Write the fourth-order Taylor polynominal for the function $f(x,y) = \cos(x+y)\ln(1+x^2)$ around the point $(0,0)$.

Soln:

$$\cos(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = 1 - \frac{r^2}{2!} + \frac{r^4}{4!} - \ldots \text{ for } r\in R$$

$$\ln(1+s) = \sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{s^n}{n}\right) = s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4!} \text{ for } |s|<1$$

$$\cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \dots$$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \dots \quad , \ |x|<1$$

$$(\cos(x+y))(\ln(1+x^2)) = \left(1 - \frac{(x+y)^2}{2} + \frac{(x+y)^4}{24}\right)\left(x^2 - \frac{x^4}{2}\right)$$

$$= x^2 - \frac{x^4}{2} - \frac{x^2(x+y)^2}{2} \dots$$